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A NOTE ON RIEMANN'S ZETA-FUNCTION

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A note on Riemann's zeta-function

by

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ABSTRACT

    This note deals with the question whether a function of the form
$f(t) = \int_X \phi(x) \cos(t\psi(x)) \, d\mu(x)$ has infinitely many zeros.
    As an application of the technique developped it is shown that the
imaginary part of $\zeta(1+it)$ has infinitely many zeros.

0. INTRODUCTION

The subject of this note was inspired by the following observation. Let

$$(1) \qquad f(t) = \sum_{n=1}^{N} a_n \cos(t\lambda_n), \qquad (t \in \mathbb{R})$$

where all a's and λ's are *real* and different from zero. Clearly f is continuous and bounded so that we may consider the (one-sided) Laplace transform $\check{f}$ of f for s > 0:

$$(2) \qquad \check{f}(s) = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} \sum_{n=1}^{N} a_n \cos(t\lambda_n)dt =$$

$$= \sum_{n=1}^{N} a_n \int_0^\infty e^{-st} \frac{e^{it\lambda_n} + e^{-it\lambda_n}}{2} dt =$$

$$= \frac{1}{2} \sum_{n=1}^{N} a_n \left\{ \frac{1}{s - i\lambda_n} + \frac{1}{s + i\lambda_n} \right\}.$$

Differentiating k times with respect to s we obtain

$$(3) \qquad \int_0^\infty e^{-st} t^k f(t)dt = \frac{k!}{2} \sum_{n=1}^{N} a_n \left\{ \frac{1}{(s-i\lambda_n)^{k+1}} + \frac{1}{(s+i\lambda_n)^{k+1}} \right\},$$

from which it is clear that for all $k \in \mathbb{N}$

$$(4) \qquad \lim_{s \downarrow 0} \int_0^\infty e^{-st} t^k f(t)dt = \frac{k!}{2i^{k+1}} \sum_{n=1}^{N} a_n \frac{(-1)^{k+1} + 1}{\lambda_n^{k+1}}.$$

Taking k even it follows that

$$(5) \qquad \lim_{s \downarrow 0} \int_0^\infty e^{-st} t^{2m} f(t)dt = 0, \qquad (\forall m \in \mathbb{N}).$$

From (5) it is intuitively clear that f(t) must have infinitely many (real) zeros.

A formal proof may be carried out as follows.

Suppose that (the even function) $f(t)$ has only finitely many zeros (possibly none). Then $f(t)$ is eventually of fixed sign and without loss of generality we may assume that

$$(6) \qquad f(t) > 0 \quad \text{for} \quad t \geq t_0 > 0.$$

Define

$$(7) \qquad M = \max_{0 \leq t \leq t_0} |f(t)|$$

and

$$(8) \qquad \mu = \min_{t_0 \leq t \leq st_0+1} |f(t)|$$

so that $\mu > 0$.

Now choose $m \in \mathbb{N}$ such that

$$(9) \qquad \mu\left(1 + \frac{1}{t_0}\right)^{2m+1} - (\mu+M) > 0.$$

Then, for any $s > 0$, we have

$$(10) \qquad \int_0^\infty e^{-st} t^{2m} f(t)\,dt = \left\{ \int_0^{t_0} + \int_{t_0}^{t_0+1} + \int_{t_0+1}^\infty \right\} e^{-st} t^{2m} f(t)\,dt >$$

$$> -M \int_0^{t_0} e^{-st} t^{2m}\,dt + \mu \int_{t_0}^{t_0+1} e^{-st} t^{2m}\,dt$$

and letting s tend to zero it follows that

$$(11) \qquad 0 \geq \mu \int_{t_0}^{t_0+1} t^{2m}\,dt - M \int_0^{t_0} t^{2m}\,dt = \mu\,\frac{(t_0+1)^{2m+1} - t_0^{2m+1}}{2m+1} - M\frac{t_0^{2m+1}}{2m+1} =$$

$$= \frac{t_0^{2m+1}}{2m+1}\left\{\mu\left(1 + \frac{1}{t_0}\right)^{2m+1} - (\mu+M)\right\} > 0$$

which is a contradiction, proving that f(t) must have infinitely many zeros.

In section 1 it will be shown that the technique illustrated above may be generalized considerably and in section 2 we will describe some consequences, the main one being the remarkable fact that the imaginary part of $\zeta(1 + it)$, $(t \in \mathbb{R})$, has infinitely many zeros.

SECTION 1

THEOREM 1. *Let* X *be a locally compact Hausdorff space equipped with a (non-negative) regular measure* μ *such that* X *is sigma- finite with respect to* μ *and* $\mu(K) < \infty$ *for any compact subset* K *of* X.

Let the μ-measurable functions $\phi, \psi : X \to \mathbb{R}$ *be such that*

(i) $\phi \in L^1(X, \mu)$

(ii) ψ *is bounded on compact subsets of* X

(iii) *there exists a μ-measurable subset* S *of* X *such that*

(12) $\displaystyle\int_X |\phi| \; d\mu = \int_S |\phi| \; d\mu$

(13) $\displaystyle\inf_{x \in S} |\psi(x)| > 0$

Then the function f$: \mathbb{R} \to \mathbb{R}$, *defined by*

(14) $\displaystyle f(t) = \int_X \phi(x) \, \cos(t\psi(x)) \, d\mu(x), \qquad (t \in \mathbb{R})$

has infinitely many (real) zeros.

PROOF. First of all we note that f is well defined indeed. Next, we observe that f is bounded and continuous on $\mathbb{R}$. It is clear that

(15) $\displaystyle |f(t)| \leq \int_X |\phi| \; d\mu$

so that

(16) $\qquad \|f\|_\infty \leq \|\phi\|_1 < \infty$

In order to see that f is continuous we recall that the set of all continuous complex valued functions on X, having compact support, is dense in $L^1(X,\mu)$. Given $\varepsilon > 0$ we may therefore choose a continuous $\phi^*: X \to \mathbb{C}$ with compact support K such that

(17) $\qquad \displaystyle\int_X |\phi - \phi^*|\, d\mu < \varepsilon.$

Let

(18) $\qquad M = \displaystyle\sup_{x \in K} |\psi(x)|.$

In order to prove the continuity of f we may just as well prove that $\tilde{f}: \mathbb{R} \to \mathbb{C}$ defined by

(19) $\qquad \tilde{f}(t) = \displaystyle\int_X \phi(x)\, e^{it\psi(x)} d\mu(x), \qquad (t \in \mathbb{R})$

is continuous on $\mathbb{R}$. For any $t_1, t_2 \in \mathbb{R}$ we have

(20) $\qquad |\tilde{f}(t_1) - \tilde{f}(t_2)| = \left| \displaystyle\int_X \phi(x)\, \{e^{it_1\psi(x)} - e^{it_2\psi(x)}\} d\mu(x) \right| =$

$\qquad = \left| \displaystyle\int_X \{\phi(x) - \phi^*(x)\}\, \{e^{it_1\psi(x)} - e^{it_2\psi(x)}\} d\mu(x)\ +\right.$

$\qquad \left. + \displaystyle\int_X \phi^*(x)\, \{e^{it_1\psi(x)} - e^{it_2\psi(x)}\} d\mu(x) \right| \leq$

$\qquad \leq 2 \displaystyle\int_X |\phi - \phi^*| d\mu + \int_K |\phi^*(x)|\, |e^{i(t_1-t_2)\psi(x)} - 1| d\mu(x) \leq$

$\qquad \leq 2 \cdot \varepsilon + |t_1 - t_2|\, M \cdot e^{|t_1-t_2|M} \displaystyle\int_K |\phi^*| d\mu$

from which it is clear that $\tilde{f}$, and hence f, is (uniformly) continuous on
$\mathbb{R}$. Since f is continuous and bounded we may consider the Laplace transform
$\breve{f}$ of f for s > 0:

(21) $\qquad \breve{f}(s) = \int\limits_0^\infty e^{-st} f(t)dt = \int\limits_0^\infty e^{-st} \int\limits_X \phi(x) \cos(t\psi(x)) \, d\mu(x) \, dt.$

The conditions in our theorem imply (c.f. [1]) that we may apply Fubini's
theorem to the repeated integral in (21) so that

(22) $\qquad \int\limits_0^\infty e^{-st} f(t)dt = \int\limits_X \phi(x) \int\limits_0^\infty e^{-st} \cos(t\psi(x))dt \, d\mu(x) =$

$\qquad\qquad = \int\limits_X \phi(x) \int\limits_0^\infty e^{-st} \dfrac{e^{it\psi(x)} + e^{-it\psi(x)}}{2} \, dt \, d\mu(x) =$

$\qquad\qquad = \tfrac{1}{2} \int\limits_X \phi(x) \{\dfrac{1}{s - i\psi(x)} + \dfrac{1}{s + i\psi(x)}\} d\mu(x) =$

$\qquad\qquad = \tfrac{1}{2} \int\limits_S \phi(x) \{\dfrac{1}{s - i\psi(x)} + \dfrac{1}{s + i\psi(x)}\} d\mu(x).$

As before, differentiate k times with respect to s in order to obtain

(23) $\qquad \int\limits_0^\infty e^{-st} t^k f(t)dt = \dfrac{k!}{2} \int\limits_S \phi(x) \{\dfrac{1}{(s-i\psi(x))^{k+1}} + \dfrac{1}{(s+i\psi(x))^{k+1}}\} d\mu(x).$

Taking limits for s ↓ 0 it follows that

(24) $\qquad \lim\limits_{s\downarrow 0} \int\limits_0^\infty e^{-st} t^k f(t)dt = \dfrac{k!}{2i^{k+1}} \int\limits_S \phi(x) \dfrac{(-1)^{k+1} + 1}{(\psi(x))^{k+1}} \, d\mu(x),$

for all k $\in \mathbb{N}$. Hence, choosing k even we find that

(25) $\qquad \lim\limits_{s\downarrow 0} \int\limits_0^\infty e^{-st} t^{2m} f(t)dt = 0, \qquad (\forall m \in \mathbb{N}).$

Similarly as in the introduction, it follows that f(t) has infinitely many
(real) zeros. $\square$

REMARK. Theorem 1 also holds true if in the definition of f we replace cos by sin. The proof is virtually the same, the only difference being that we have to choose k odd instead of even.

The theorem also holds if we replace condition (iii) by the following one

(26) (iii)* $\dfrac{\phi}{\psi^k} \in L^1(X,\mu)$, $(\forall k \in \mathbb{N})$.

The details of the proof are left to the reader.

SECTION 2

In this section we will first apply the technique illustrated in sections 0 and 1 to the imaginary part of $\zeta(1 + it)$, where ζ denotes Riemann's zeta-function.

THEOREM 2. *Let*

(27) $I(t) = \operatorname{Im} \zeta(1 + it) = \dfrac{\zeta(1 + it) - \zeta(1 - it)}{2i}$, $(t>0)$.

Then $I(t)$ *has infinitely many (real) zeros.*

PROOF. For $\sigma > 1$ we define $I_\sigma : \mathbb{R}^+ \to \mathbb{R}$ by

(28) $I_\sigma(t) = \operatorname{Im} \zeta(\sigma + it) = \dfrac{\zeta(\sigma + it) - \zeta(\sigma - it)}{2i}$, $(t \in \mathbb{R}^+)$.

Then $I_\sigma(t)$, as a function of t, is continuous on $\mathbb{R}^+$ and since $|I_\sigma(t)| \le \zeta(\sigma)$ for all $t \in \mathbb{R}^+$, $I_\sigma(t)$ is also bounded. We consider the Laplace transform $\breve{I}_\sigma$ of I_σ for s > 0:

(29) $\breve{I}_\sigma(s) = \displaystyle\int_0^\infty e^{-st} I_\sigma(t)dt = \int_0^\infty e^{-st} \{ \sum_{n=2}^\infty \dfrac{1}{n^\sigma} \dfrac{n^{-it} - n^{it}}{2i} \}dt =$

$= \dfrac{1}{2i} \displaystyle\sum_{n=2}^\infty \dfrac{1}{n^\sigma} \{ \dfrac{1}{s + i \log n} - \dfrac{1}{s - i \log n} \}$, $(s>0)$.

Differentiation with respect to s yields

(30) $\displaystyle \int_0^\infty e^{-st}t\ I_\sigma(t)dt = \frac{1}{2i} \sum_{n=2}^\infty \frac{1}{n^\sigma} \{\frac{1}{(s+i\log n)^2} - \frac{1}{(s-i\log n)^2}\}$.

Observing that $(z-1)\zeta(z)$ is an entire function and that (c.f. [2] p.42)

(31) $\zeta(\sigma + it) = O(\log t)$,

uniformly in the region $\sigma \geq 1$, $t \geq 2$, it is readily seen (using Lebesgue's dominated convergence theorem and the uniform convergence of the series in (30) for $\sigma \geq 1$ and $s \in \mathbb{R}$) that

(32) $\displaystyle \lim_{\sigma \downarrow 1} \int_0^\infty e^{-st}t\ I_\sigma(t)dt = \int_0^\infty e^{-st}t\ I(t)dt =$

$\displaystyle = \frac{1}{2i} \sum_{n=2}^\infty \frac{1}{n} \{\frac{1}{(s+i\log n)^2} - \frac{1}{(s-i\log n)^2}\}$.

Differentiating (k-1) times with respect to s we obtain for $s > 0$ and $k \in \mathbb{N}$

(33) $\displaystyle \int_0^\infty e^{-st}t^k\ I(t)dt = \frac{k!}{2i} \sum_{n=2}^\infty \frac{1}{n} \{\frac{1}{(s+i\log n)^{k+1}} - \frac{1}{(s-i\log n)^{k+1}}\}$,

Similarly, as before, we find that

(34) $\displaystyle \lim_{s \downarrow 0} \int_0^\infty e^{-st}t^k\ I(t)dt = \frac{k!}{2i^k} \sum_{n=2}^\infty \frac{1}{n} \frac{(-1)^{k+1}-1}{(\log n)^{k+1}}$

and taking k odd it follows that

(35) $\displaystyle \lim_{s \downarrow 0} \int_0^\infty e^{-st}t^{2m+1}\ I(t)dt = 0$, $(\forall m \in \mathbb{N})$.

Similarly as before, we arrive at the remarkable result that $I(t) = \text{Im}\ \zeta(1+it)$ has infinitely many (real) zeros. $\square$

REMARKS. In a similar manner it can be shown that Re $\zeta(1+it)$ takes the value 1 infinitely many times.

Also, by the same method, one may show that for any fixed $\sigma > 0$ the function Im $\eta(\sigma+it)$ (where $\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$ for Re $s > 0$) has infinitely many real zeros.

As a direct application of theorem 1 we have the following

THEOREM 3. *Let $\phi: \mathbb{R} \to \mathbb{R}$ be an even (Lebesgue) integrable function which vanishes in a neighborhood of zero. Then the Fourier transform $\hat{\phi}$ of ϕ has infinitely many (real) zeros.*

PROOF. Observe that

$$(36) \qquad \hat{\phi}(t) = \int_{\mathbb{R}} e^{itx} \phi(x)\,dx = \int_{\mathbb{R}} \phi(x)\,(\cos tx + i \sin tx)\,dx =$$

$$= \int_{\mathbb{R}} \phi(x)\,\cos tx\,dx = 2 \int_{a}^{\infty} \phi(x)\,\cos tx\,dx, \quad \text{for some } a > 0.$$

It is easily verified that all conditions of theorem 1 are satisfied so that $\hat{\phi}(t)$ has infinitely many zeros. $\square$

REMARK. Note that theorem 3 may by generalized as indicated in the second remark following the proof of theorem 1.

REFERENCES

[1] RUDIN, W., *Real and complex analysis*, McGraw-Hill, 1966.

[2] TITCHMARSH, E.C., *The theory of the Riemann zeta-function*, Oxford, 1951.